ΜΑΘΗΜΑΤΙΚΑ ΙΙ

Ηλεκτρολόγων Μηχανικών και Τεχνολογίας Υπολογιστών Παν.Πατρών

ΛΥΜΕΝΕΣ ΑΣΚΗΣΕΙΣ

ΧΕΙΡΟΓΡΑΦΕΣ ΣΗΜΕΙΩΣΕΙΣ

ΕΠΙΜΕΛΕΙΑ ΣΗΜΕΙΩΣΕΩΝ: Uni Student

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ΑΝΤΙ ΠΡΟΛΟΓΟΥ

Τα παρακάτω αποτελούν σημειώσεις του μαθήματος "Μαθηματικά ΙΙ" του τμήματος Ηλεκτρολόγων Μηχανικών και Τεχνολογίας Υπολογιστών της Πολυτεχνικής Σχολής του Πανεπιστημίου Πατρών. Οι σημειώσεις περιέχουν λυμένες ασκήσεις Μαθηματικών από τις παραδόσεις του μαθήματος.

ΕΠΙΜΕΛΕΙΑ ΣΗΜΕΙΩΣΕΩΝ: Uni Student

ΜΑΘΗΜΑΤΙΚΑ ΙΙ

ΑΣΚΗΣΕΙΣ

1) Να δειχτεί ότι η f είναι συνεχής σε κάθε σημείο του άξονα Oy.

$$f(x,y) = \begin{cases} \dfrac{1-\cos\sqrt{xy}}{x} & , \; xy \geq 0 \\[2ex] \dfrac{y}{2} & , \; x = 0 \end{cases}$$

Λύση

Για τυχαίο σημείο στον άξονα Oy, θα ισχύει $f(0,y_0) = \frac{y_0}{2}$. (1)

$$\lim_{\substack{x\to 0\\ y\to y_0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to y_0}} \frac{1-\cos\sqrt{xy}}{x} = \lim_{\substack{x\to 0\\ y\to y_0}} \frac{2\sin^2\frac{\sqrt{xy}}{2}}{x} =$$

Έχω
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1 \Leftrightarrow \cos 2x - 1 = -2\sin^2 x$$

$$= \lim_{\substack{x\to 0\\ y\to y_0}} \frac{2\sin^2\frac{\sqrt{xy}}{2}}{\left(\frac{\sqrt{xy}}{2}\right)^2 \cdot \frac{4}{y}} = \frac{1}{2} \lim_{\substack{x\to 0\\ y\to y_0}} \left[\underbrace{\left(\frac{\sin\frac{\sqrt{xy}}{2}}{\frac{\sqrt{xy}}{2}} \right)^2}_{\to 1} \cdot y \right] =$$

$$= \frac{1}{2} \cdot y_0 = \frac{y_0}{2} = f(0,y_0)$$

δηλαδή η $f(x,y)$ είναι συνεχής σε κάθε σημείο του άξονα Oy.

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2) Να δειχτεί ότι ισχύει $\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0$ για τη συνάρτηση:
$f(x,y) = \sin(x-y) + \ln(x+y)$.

Λύση

$$\frac{\partial f}{\partial x} = \cos(x-y) + \frac{1}{x+y} \quad , \quad \frac{\partial f}{\partial y} = -\cos(x-y) + \frac{1}{x+y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\cos(x-y) + \frac{1}{x+y}\right] = -\sin(x-y) + \frac{(1)'(x+y) - 1(x+y)'}{(x+y)^2}$$

$$\Leftrightarrow \frac{\partial^2 f}{\partial x^2} = -\sin(x-y) - \frac{1}{(x+y)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[-\cos(x-y) + \frac{1}{x+y}\right] = -\sin(x-y) + \frac{-1}{(x+y)^2}$$

άρα $$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = -\sin(x-y) - \frac{1}{(x+y)^2} + \sin(x-y) + \frac{1}{(x+y)^2} = 0.$$

3) Να δειχτεί ότι ισχύει $\frac{\partial^2 f}{\partial x^2} - 9\frac{\partial^2 f}{\partial y^2} = 0$ για τη συνάρτηση: $f(x,y) = (y+3x)^{1/2} - (y-3x)^2$.

Λύση

$$\frac{\partial f}{\partial x} = \frac{1}{2}(y+3x)^{-1/2} \cdot 3 - 2(y-3x)\cdot(-3) = \frac{3}{2}(y+3x)^{-1/2} + 6(y-3x)$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(y+3x)^{-1/2} - 2(y-3x)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{3}{2}(y+3x)^{-1/2} + 6(y-3x)\right]$$

$$\Leftrightarrow \frac{\partial^2 f}{\partial x^2} = -\frac{3}{4}(y+3x)^{-3/2} \cdot 3 + 6(-3) = -\frac{9}{4}(y+3x)^{-3/2} - 18$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1}{2}(y+3x)^{-1/2} - 2(y-3x)\right]$$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = -\frac{1}{4}(y+3x)^{-3/2} - 2$$

άρα

$$\frac{\partial^2 f}{\partial x^2} - 9\frac{\partial^2 f}{\partial y^2} = -\frac{9}{4}(y+3x)^{-3/2} - 18 + \frac{9}{4}(y+3x)^{-3/2} + 18 = 0.$$

4) Να αποδείξετε ότι αν η συνάρτηση $f(x,y)$ είναι αρμονική, τότε και η συνάρτηση f_y είναι αρμονική. ②

Λύση

Έστω $g(x,y) = f_y$. Θα πρέπει να ισχύει: $g_{xx} + g_{yy} = 0$ για να είναι η f_y αρμονική συνάρτηση.

$g_x = f_{yx} = \frac{\partial}{\partial x} f_y$ $\qquad$ $g_{xx} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right] = f_{yxx}$

$g_y = f_{yy} = \frac{\partial}{\partial y} f_y$ $\qquad$ $g_{yy} = \frac{\partial}{\partial y}\frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right]$

δηλαδή $g_{yy} = \frac{\partial}{\partial y} f_{yy}$

και $g_{xx} = f_{yxx} = f_{xxy} = \frac{\partial}{\partial y} f_{xx}$ (από θεώρημα Schwarz)

άρα $g_{xx} + g_{yy} = \frac{\partial}{\partial y} f_{xx} + \frac{\partial}{\partial y} f_{yy} = \frac{\partial}{\partial y}\left(f_{xx} + f_{yy}\right)$

Όμως η f είναι αρμονική, δηλαδή $f_{xx} + f_{yy} = 0$, οπότε:

$g_{xx} + g_{yy} = \frac{\partial}{\partial y}\left(f_{xx} + f_{yy}\right) = 0$, δηλαδή η f_y είναι αρμονική.

5) Να αποδείξετε ότι αν η συνάρτηση $f(x,y)$ είναι αρμονική, τότε και η συνάρτηση $x\cdot f_y - y\cdot f_x$ είναι αρμονική.

Λύση

Έστω $g(x,y) = x\cdot f_y - y\cdot f_x$. Θα πρέπει να ισχύει: $g_{xx} + g_{yy} = 0$ για να είναι η $x f_y - y f_x$ αρμονική συνάρτηση.

$g_x = (x)'\cdot f_y + x\cdot\frac{\partial}{\partial x} f_y - y\cdot\frac{\partial}{\partial x} f_x = f_y + x\cdot f_{yx} - y\cdot f_{xx}$

$g_y = x\cdot\frac{\partial}{\partial y} f_y - (y)'\cdot f_x - y\cdot\frac{\partial}{\partial y} f_x = x\cdot f_{yy} - f_x - y\cdot f_{xy}$

$g_{xx} = \frac{\partial}{\partial x}\left[f_y + x\cdot f_{yx} - y\cdot f_{xx}\right] = \frac{\partial}{\partial x} f_y + f_{yx} + x\cdot\frac{\partial}{\partial x} f_{yx} - y\frac{\partial}{\partial x} f_{xx}$

$\Rightarrow g_{xx} = f_{yx} + f_{yx} + x\cdot f_{yxx} - y\cdot f_{xxx}$

$g_{yy} = \frac{\partial}{\partial y}\left[x f_{yy} - f_x - y\cdot f_{xy}\right] = x\cdot\frac{\partial}{\partial y} f_{yy} - \frac{\partial}{\partial y} f_x - f_{xy} - y\frac{\partial}{\partial y} f_{xy}$

$\ldots - y \cdot f_{xxx} + x \cdot f_{yyy} \ldots$

$g_{xx} + g_{yy} = 2f_{yx} + x \cdot f_{yxx} - y \cdot f_{xxx} + x \cdot f_{yyy} \ldots$

Από θεώρημα Schwarz ισχύει ότι: $f_{yx} = f_{xy}$, άρα:

$g_{yy} = x \cdot f_{yxx} - y \cdot f_{xxx} + x \cdot f_{yyy} - y \cdot f_{xyy}$

Από θεώρημα Schwarz πάλι, έχω:

$= f_{xxy} = \frac{\partial}{\partial y} f_{xx}$

$f_{xyy} = f_{yyx} = \frac{\partial}{\partial x} f_{yy}$, δηλαδή:

$g_{yy} = x\left[\frac{\partial}{\partial y} f_{xx} + \frac{\partial}{\partial y} f_{yy}\right] - y\left[\frac{\partial}{\partial x} f_{xx} + \frac{\partial}{\partial x} f_{yy}\right]$

$g_{xx} + g_{yy} = x \frac{\partial}{\partial y}(f_{xx} + f_{yy}) - y \frac{\partial}{\partial x}(f_{xx} + f_{yy})$

ς, η f είναι αρμονική, δηλαδή $f_{xx} + f_{yy} = 0$, άρα:

$g_{xx} + g_{yy} = \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)(f_{xx} + f_{yy}) = 0$, δηλαδή

$x f_y - y f_x$ είναι αρμονική συνάρτηση.

) Να εξεταστεί αν η παράσταση $(2x\cos y + 2)dx + (-x^2 \cdot \sin y + 6y)dy$ είναι ακριβές διαφορικό και να βρεθεί η συνάρτηση δυναμικού.

Λύση

α' τρόπος

Έστω $P(x,y) = 2x\cos y + 2$
$Q(x,y) = -x^2 \cdot \sin y + 6y$
πρέπει $P_y = Q_x$ για να είναι η παρά... ακριβές διαφορικό

$P_y = -2x \cdot \sin y$
$Q_x = -2x \cdot \sin y$
$P_y = Q_x$, άρα η παράσταση είναι ακριβές δια...

Συνάρτηση δυναμικού

$\ldots \int^{y} \ldots + 6y)\,dy = \int^{x}_{\ldots}(2x \cdot \cos y + 2)\,dx + \int_0^{y} \ldots$

$f(x,y) = \left[\cos y \cdot \frac{x}{\cancel{2}} + 2x\right]_0 + \left[3y^2\right]_0 + c$

$\Leftrightarrow f(x,y) = \cos y \cdot x^2 + 2x + 3y^2 + c$

β' τρόπος

Έστω $f(x,y)$ όπου η παράσταση $(2x \cdot \cos y + 2)dx + (-x^2 \cdot \sin y + 6y)dy$ είναι το διαφορικό της, δηλαδή η παράσταση ισούται με df.

Πρέπει: $f_x = 2x \cdot \cos y + 2 \quad (1)$

$f_y = -x^2 \cdot \sin y + 6y \quad (2)$

Ολοκληρώνοντας την (1), έχω:

$\int f_x dx = \int (2x \cdot \cos y + 2)dx \Leftrightarrow f(x,y) = x^2 \cdot \cos y + 2x + c(y) \quad (3)$

όπου $c(y)$ μία ~~στα~~ συνάρτηση μόνο του y (ανεξάρτητη του x).

Παραγωγίζοντας την (3) ως προς y, έχω

$$\frac{\partial f}{\partial y} = -x^2 \cdot \sin y + \frac{\partial c}{\partial y} \quad (4)$$

Όμως, $\frac{\partial f}{\partial y} = f_y$ και $\frac{\partial c}{\partial y} = \frac{dc}{dy}$, αφού η $c(y)$ είναι συνάρτηση μιας μεταβλητής

Άρα: (2),(4): $-x^2 \cdot \sin y + 6y = -x^2 \cdot \sin y + \frac{dc}{dy} \Leftrightarrow \frac{dc}{dy} = 6y \Leftrightarrow dc = 6y\,dy$

$\Rightarrow \int \frac{dc}{\cancel{dy}} = \int 6y\,\cancel{d}y \Leftrightarrow c(y) = 3y^2 + c_1$

δηλαδή $f(x,y) = x^2 \cdot \cos y + 2x + 3y^2 + c$

7) Να εξεταστεί αν η παράσταση $(y+z)dx+(z+x)dy+(x+y)dz$ είναι τέλειο διαφορικό και να βρεθεί η συνάρτηση δυναμικού.

<u>Λύση</u>

Έστω
$$\left.\begin{aligned} P(x,y,z) &= y+z \\ Q(x,y,z) &= z+x \\ R(x,y,z) &= x+y \end{aligned}\right\}$$
Για να είναι η παράσταση ακριβές διαφορικό, πρέπει να ισχύουν οι σχέσεις:
$$P_y = Q_x$$
$$Q_z = R_y$$
$$R_x = P_z$$

$$\left.\begin{aligned} P_y &= 1 \\ Q_x &= 1 \end{aligned}\right\} P_y = Q_x \qquad \left.\begin{aligned} Q_z &= 1 \\ R_y &= 1 \end{aligned}\right\} Q_z = R_y \qquad \left.\begin{aligned} R_x &= 1 \\ P_z &= 1 \end{aligned}\right\} R_x = P_z$$

άρα η παραπάνω παράσταση είναι ακριβές διαφορικό.

$$f(x,y,z) = \int_0^x (y+z)dx + \int_0^y (z+0)dy + \int_0^z (0+0)dz + c$$

$$\Leftrightarrow f(x,y,z) = \left[(y+z)x\right]_0^x + \left[zy\right]_0^y + c$$

$$\Leftrightarrow f(x,y,z) = yx + zx + zy + c$$

8) Να υπολογιστεί η παράγωγος α' τάξης της σύνθετης συνάρτησης $f(x,y) = \ln(y^2-x^2)$ όπου $x=\sin t$, $y=\cos t$ για $t=\pi/8$.

<u>Λύση</u>

$$\frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = -\sin t, \quad f_x = \frac{-2x}{y^2-x^2}, \quad f_y = \frac{2y}{y^2-x^2}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} \Leftrightarrow \frac{df}{dt} = \frac{-2x}{y^2-x^2}\cdot\cos t + \frac{2y}{y^2-x^2}\cdot(-\sin t)$$

$$\Leftrightarrow \frac{df}{dt} = \frac{-2(x\cdot\cos t + y\cdot\sin t)}{y^2-x^2} \Leftrightarrow \frac{df}{dt} = \frac{-2(\sin t\cdot\cos t + \cos t\cdot\sin t)}{\cos^2 t - \sin^2 t} = \frac{-2\sin 2t}{\cos 2t}$$

$$\left.\frac{df}{dt}\right|_{t=\pi/8} = \frac{-2\sin 2\frac{\pi}{8}}{\cos 2\frac{\pi}{8}} = \frac{-2\sin \pi/4}{\cos \pi/4} = \frac{-2\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{-4\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}} = -2.$$

9) Να βρεθούν τα: α) $\frac{\partial^2 f}{\partial v^2}$, β) $\frac{\partial^2 f}{\partial u \partial v}$ της συνάρτησης $w=f(x,y)$ (4)

όπου $x=x(u,v)$ και $y=y(u,v)$.

Λύση

$$\frac{\partial^2 f}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right) = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right) =$$

$$= \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right) =$$

$$= \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial v} + \frac{\partial f}{\partial x}\frac{\partial}{\partial v}\left(\frac{\partial x}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial}{\partial v}\left(\frac{\partial y}{\partial v}\right) =$$

$$= \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial v}\right)\frac{\partial x}{\partial v} + \frac{\partial^2 x}{\partial v^2}\frac{\partial f}{\partial x} + \cdots$$

$$\cdots + \left(\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial v} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial v}\right)\frac{\partial y}{\partial v} + \frac{\partial^2 y}{\partial v^2}\frac{\partial f}{\partial y} =$$

$$= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial v}\right)^2 + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial v}\frac{\partial x}{\partial v} + \frac{\partial^2 x}{\partial v^2}\frac{\partial f}{\partial x} + \cdots$$

$$\cdots + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial v}\right)^2 + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial v}\frac{\partial y}{\partial v} + \frac{\partial^2 y}{\partial v^2}\frac{\partial f}{\partial y} =$$

$$= \underbrace{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial v}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial y}{\partial v}\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial v}\right)^2}_{(2)} + \frac{\partial^2 x}{\partial v^2}\frac{\partial f}{\partial x} + \frac{\partial^2 y}{\partial v^2}\frac{\partial f}{\partial y} =$$

$$= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right)^{(2)} + \frac{\partial^2 x}{\partial v^2}\frac{\partial f}{\partial x} + \frac{\partial^2 y}{\partial v^2}\frac{\partial f}{\partial y}$$

Δηλαδή

$$f_{vv} = \frac{\partial^2 f}{\partial v^2} = (f_x \cdot x_v + f_y\, y_v)^{(2)} + f_x \cdot x_{vv} + f_y \cdot y_{vv}$$

β) $$\frac{\partial^2 f}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right) =$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}\right) + \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right) =$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial v} + \frac{\partial f}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial v}\right) + \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right)$$

$$= \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u}\right)\frac{\partial x}{\partial v} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial u \partial v} + \cdots$$

$$\cdots + \left(\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u}\right)\frac{\partial y}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial u \partial v} =$$

$$= \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial u \partial v} + \cdots$$

$$\cdots + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial u \partial v}$$

Δηλαδή

$$f_{uv} = \frac{\partial^2 f}{\partial u \partial v} = f_{xx}\cdot x_u \cdot x_v + f_{xy}\left(y_u \cdot x_v + x_u \cdot y_v\right) + f_x \cdot x_{vu} + f_{yy}\cdot y_u \cdot y_v + f_y \cdot y_{vu}$$

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10) Έστω $f(u,v,w) = u^2 v w$ με $u(x,y) = x+y$, $v(x,y) = x-y$, $w(x,y) = xy$.
Να βρεθεί το ολικό διαφορικό β' τάξης.

Λύση

$$d^2 f = d(df) = d\left(f_u du + f_v dv + f_w dw\right) = d\left(f_u du\right) + d\left(f_v dv\right) + d\left(f_w dw\right) =$$

$$= d(f_u)du + f_u d(du) + d(f_v)dv + f_v d(dv) + d(f_w)dw + f_w d(dw) =$$

$$= \left(f_{uu} du + f_{uv} dv + f_{uw} dw\right)du + f_u d^2 u + \left(f_{vu} du + f_{vv} dv + f_{vw} dw\right)dv + \cdots$$

$$\cdots + f_v d^2 v + \left(f_{wu} du + f_{wv} dv + f_{ww} dw\right)dw + f_w d^2 w =$$

$$= f_{uu}(du)^2 + f_{uv}\, dv\, du + f_{uw}\, dw\, du + f_u d^2 u + f_{vu}\, du\, dv + f_{vv}(dv)^2 + f_{vw}\, dw\, dv +$$

$$+ f_v d^2 v + f_{wu}\, du\, dw + f_{wv}\, dv\, dw + f_{ww}(dw)^2 + f_w d^2 w =$$

$$= \underbrace{f_{uu}(du)^2 + 2 f_{uv}\, dv\, du + 2 f_{uw}\, dw\, du + 2 f_{wv}\, dv\, dw + f_{vv}(dv)^2 + f_{ww}(dw)^2}_{} +$$

$$+ f_u d^2 u + f_v d^2 v + f_w d^2 w$$

$$=\left(f_u du + f_v dv + f_w dw\right)^{(2)} + f_u d^2u + f_v d^2v + f_w d^2w \quad (1)$$

Έχω

$f_u = 2uvw$, $du = dx+dy \Rightarrow d^2u = d(du) = d(dx+dy) = d^2x + d^2y$

$f_v = u^2w$, $dv = dx-dy \Rightarrow d^2v = d(dv) = d(dx-dy) = d^2x - d^2y$

$f_w = u^2v$, $dw = ydx + xdy \Rightarrow d^2w = d(dw) = d(ydx+xdy) = d(ydx) + d(xdy)$

$$\Leftrightarrow d^2w = dydx + yd^2x + dxdy + xd^2y = yd^2x + xd^2y + 2\,dxdy$$

άρα: $d^2f = 2v\cdot w(dx+dy)^2 + 2\cdot 2uw(dx-dy)(dx+dy) + 2\cdot 2uv(ydx+xdy)$

$(dx+dy) + 2u^2(dx-dy)(ydx+xdy) + 0\cdot(dv)^2 + 0\cdot(dw)^2 + 2uvw(d^2x+d^2y) +$

$+ u^2w(d^2x-d^2y) + u^2v(yd^2x+xd^2y+2dxdy)$

$\Leftrightarrow d^2f = 2vw\left[(dx)^2 + 2dxdy + (dy)^2\right] + 4uw\left[(dx)^2 - (dy)^2\right] + 4uv(ydx+xdy)(dx+dy)$

$+ 2u^2(dx-dy)(ydx+xdy) + 2uvw(d^2x+d^2y) + u^2w(d^2x-d^2y) + \ldots$

$\ldots + u^2v(yd^2x+xd^2y+2dxdy)$

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11) Δίνεται $z = e^{uv}$, όπου $u = \ln(x+y)$ και $v = \arctan\frac{x}{y}$. Να βρείτε τα $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, dz.

Λύση

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad (1)$$

άρα: $\frac{\partial z}{\partial u} = e^{uv}\cdot v$, $\frac{\partial z}{\partial v} = e^{uv}\cdot u$, $\frac{\partial u}{\partial x} = \frac{1}{x+y}$, $\frac{\partial v}{\partial x} = \frac{\frac{1}{y}}{1+\left(\frac{x}{y}\right)^2}$

$$(1) \Rightarrow \frac{\partial z}{\partial x} = \frac{e^{uv}\cdot v}{x+y} + e^{uv}u\cdot\frac{\frac{1}{y}}{1+\left(\frac{x}{y}\right)^2} = \frac{e^{uv}\cdot v}{x+y} + e^{uv}\cdot u\,\frac{\frac{1}{y}}{\frac{x^2+y^2}{y^2}}$$

$$\Leftrightarrow \frac{\partial z}{\partial x} = \frac{e^{uv}\cdot v}{x+y} + \frac{e^{uv}\cdot u\cdot y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \quad (2)$$

άρα : $\frac{\partial z}{\partial u} = e^{uv}\cdot v$, $\frac{\partial u}{\partial y} = \frac{1}{x+y}$, $\frac{\partial z}{\partial v} = e^{uv}\cdot u$, $\frac{\partial v}{\partial y} = \frac{\frac{-x}{y^2}}{1+\left(\frac{x}{y}\right)^2}$

$$(2) \Rightarrow \frac{\partial z}{\partial y} = \frac{e^{uv}\cdot v}{x+y} + e^{uv}\cdot u \cdot \frac{\frac{-x}{y^2}}{1+\left(\frac{x}{y}\right)^2} = \frac{e^{uv}\cdot v}{x+y} + e^{uv}\cdot u \frac{\frac{-x}{\cancel{y^2}}}{\frac{x^2+y^2}{\cancel{y^2}}}$$

$$\Leftrightarrow \frac{\partial z}{\partial y} = \frac{e^{uv}\cdot v}{x+y} - \frac{e^{uv}\cdot u \cdot x}{x^2+y^2}$$

$$dz = z_x dx + z_y dy = z_u du + z_v dv \quad (3)$$

άρα: $z_u = \frac{\partial z}{\partial u} = e^{uv}\cdot v$, $z_v = \frac{\partial z}{\partial v} = e^{uv}\cdot u$

$$du = u_x dx + u_y dy = \frac{dx}{x+y} + \frac{dy}{x+y}$$

$$dv = v_x dx + v_y dy = \frac{y\,dx}{x^2+y^2} - \frac{x\,dy}{x^2+y^2}$$

$$(3) \Rightarrow dz = e^{uv}\cdot v\left(\frac{dx}{x+y} + \frac{dy}{x+y}\right) + e^{uv}\cdot u\left(\frac{y\,dx}{x^2+y^2} - \frac{x\,dy}{x^2+y^2}\right)$$

$$\Leftrightarrow dz = \left(\frac{e^{uv}\cdot v}{x+y} + \frac{e^{uv}\cdot u y}{x^2+y^2}\right)dx + \left(\frac{e^{uv}\cdot v}{x+y} - \frac{e^{uv}\cdot u x}{x^2+y^2}\right)dy$$

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12) Να οριστεί κατάλληλα η $f(x,y,z)$ στο $(0,0,0)$ ώστε να είναι συνεχής σε αυτό

$f(x,y,z) = \frac{1-\cos\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}$.

Λύση

$$f(x,y,z) = \frac{1-\cos\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2} \Rightarrow \lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = \lim_{(x,y,z)\to(0,0,0)} \frac{1-\cos\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2} =$$

$\left\{ \cos 2x = 1-2\sin^2 x \right.$ άρα $1-\cos 2x = 2\sin^2 x$

$$= \lim_{\substack{x\to0\\y\to0\\z\to0}} \frac{2\sin^2\frac{\sqrt{x^2+y^2+z^2}}{2}}{\left(\frac{\sqrt{x^2+y^2+z^2}}{2}\right)^2\cdot 4} = \frac{1}{2}\lim_{\substack{x\to0\\y\to0\\z\to0}} \underbrace{\frac{\sin^2\frac{\sqrt{x^2+y^2+z^2}}{2}}{\left(\frac{\sqrt{x^2+y^2+z^2}}{2}\right)^2}}_{\to 1} = \frac{1}{2}$$

άρα
$$f(x,y,z) = \begin{cases} \frac{1-\cos\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}, & (x,y,z)\neq(0,0,0) \\ 1/2, & (x,y,z)=(0,0,0) \end{cases}$$

$$=\left(f_u du + f_v dv + f_w dw\right)^{(2)} + f_u d^2u + f_v d^2v + f_w d^2w \qquad (1)$$

Έχω

$f_u = 2uvw \quad , \quad du = dx+dy \Rightarrow d^2u = d(du) = d(dx+dy) = d^2x + d^2y$

$f_v = u^2w \quad , \quad dv = dx-dy \Rightarrow d^2v = d(dv) = d(dx-dy) = d^2x - d^2y$

$f_w = u^2v \quad , \quad dw = ydx + xdy \Rightarrow d^2w = d(dw) = d(ydx+xdy) = d(ydx) + d(xdy)$

$\Leftrightarrow d^2w = dydx + yd^2x + dxdy + xd^2y = yd^2x + xd^2y + 2\,dxdy$

άρα: $d^2f = 2v\cdot w(dx+dy)^2 + 2\cdot 2uw(dx-dy)(dx+dy) + 2\cdot 2uv(ydx+xdy)$

$(dx+dy) + 2u^2(dx-dy)(ydx+xdy) + 0\cdot(dv)^2 + 0\cdot(dw)^2 + 2uvw(d^2x+d^2y) +$

$+ u^2w(d^2x-d^2y) + u^2v(yd^2x+xd^2y+2dxdy)$

$\Leftrightarrow d^2f = 2vw\left[(dx)^2+2dxdy+(dy)^2\right] + 4uw\left[(dx)^2-(dy)^2\right] + 4uv(ydx+xdy)(dx+dy) +$

$+ 2u^2(dx-dy)(ydx+xdy) + 2uvw(d^2x+d^2y) + u^2w(d^2x-d^2y) + \ldots$

$\ldots + u^2v(yd^2x+xd^2y+2dxdy)$

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11) Δίνεται $z=e^{uv}$, όπου $u=\ln(x+y)$ και $v=\arctan\frac{x}{y}$. Να βρείτε τα $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, dz.

Λύση

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad (1)$$

άρα: $\frac{\partial z}{\partial u} = e^{uv}\cdot v$, $\frac{\partial z}{\partial v} = e^{uv}\cdot u$, $\frac{\partial u}{\partial x} = \frac{1}{x+y}$, $\frac{\partial v}{\partial x} = \frac{\frac{1}{y}}{1+\left(\frac{x}{y}\right)^2}$

$(1) \Rightarrow \frac{\partial z}{\partial x} = \frac{e^{uv}\cdot v}{x+y} + e^{uv}u\cdot\frac{\frac{1}{y}}{1+\left(\frac{x}{y}\right)^2} = \frac{e^{uv}\cdot v}{x+y} + e^{uv}\cdot u\,\frac{\frac{1}{\cancel{y}}}{\frac{x^2+y^2}{y^{\cancel{2}}}}$

$\Leftrightarrow \frac{\partial z}{\partial x} = \frac{e^{uv}\cdot v}{x+y} + \frac{e^{uv}\cdot u\cdot y}{x^2+y^2}$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \quad (2)$$

άρα : $\frac{\partial z}{\partial u} = e^{uv}\cdot v$, $\frac{\partial u}{\partial y} = \frac{1}{x+y}$, $\frac{\partial z}{\partial v} = e^{uv}\cdot u$, $\frac{\partial v}{\partial y} = \frac{\frac{-x}{y^2}}{1+\left(\frac{x}{y}\right)^2}$

$$(2) \Rightarrow \frac{\partial z}{\partial y} = \frac{e^{uv}\cdot v}{x+y} + e^{uv}\cdot u \cdot \frac{\frac{-x}{y^2}}{1+\left(\frac{x}{y}\right)^2} = \frac{e^{uv}\cdot v}{x+y} + e^{uv}\cdot u \frac{\frac{-x}{\cancel{y^2}}}{\frac{x^2+y^2}{\cancel{y^2}}}$$

$$\Leftrightarrow \frac{\partial z}{\partial y} = \frac{e^{uv}\cdot v}{x+y} - \frac{e^{uv}\cdot u\cdot x}{x^2+y^2}$$

$$dz = z_x dx + z_y dy = z_u du + z_v dv \quad (3)$$

άρα : $z_u = \frac{\partial z}{\partial u} = e^{uv}\cdot v$, $z_v = \frac{\partial z}{\partial v} = e^{uv}\cdot u$

$$du = u_x dx + u_y dy = \frac{dx}{x+y} + \frac{dy}{x+y}$$

$$dv = v_x dx + v_y dy = \frac{y\,dx}{x^2+y^2} - \frac{x\,dy}{x^2+y^2}$$

$$(3) \Rightarrow dz = e^{uv}\cdot v\left(\frac{dx}{x+y} + \frac{dy}{x+y}\right) + e^{uv}\cdot u\left(\frac{y\,dx}{x^2+y^2} - \frac{x\,dy}{x^2+y^2}\right)$$

$$\Leftrightarrow dz = \left(\frac{e^{uv}\cdot v}{x+y} + \frac{e^{uv}\cdot u y}{x^2+y^2}\right)dx + \left(\frac{e^{uv}\cdot v}{x+y} - \frac{e^{uv}\cdot u x}{x^2+y^2}\right)dy$$

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12) Να οριστεί κατάλληλα η $f(x,y,z)$ στο $(0,0,0)$ ώστε να είναι συνεχής σε αυτό.

$f(x,y,z) = \frac{1-\cos\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}$.

Λύση

$$f(x,y,z) = \frac{1-\cos\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2} \Rightarrow \lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = \lim_{(x,y,z)\to(0,0,0)} \frac{1-\cos\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2} =$$

$\left\{ \begin{array}{l} \cos 2x = \\ = 1-2\sin^2 x \\ \text{άρα} \\ 1-\cos 2x = \\ = 2\sin^2 x \end{array} \right.$

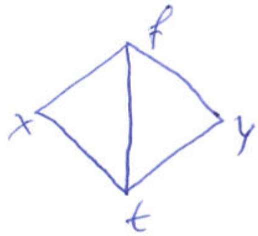
$$= \lim_{\substack{x\to 0\\ y\to 0\\ z\to 0}} \frac{2\sin^2\frac{\sqrt{x^2+y^2+z^2}}{2}}{\left(\frac{\sqrt{x^2+y^2+z^2}}{2}\right)^2\cdot 4} = \frac{1}{2}\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0}} \frac{\sin^2\frac{\sqrt{x^2+y^2+z^2}}{2}}{\frac{\sqrt{x^2+y^2+z^2}}{2}} = \frac{1}{2}$$

$$\text{άρα} \quad f(x,y,z) = \begin{cases} \frac{1-\cos\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}, & (x,y,z)\neq(0,0,0) \\ 1/2, & (x,y,z)=(0,0,0) \end{cases}$$

---

13) Δίνεται η συνάρτηση $f(x,y,t) = t^2 \cdot x \cdot \cos y + x^2 \cdot y \cdot \ln t$ όπου $x = t^2$, $y = \sin t$. Να βρεθεί το $\frac{df}{dt}$.

Λύση



$$\frac{df}{dt} = f_x \cdot x_t + f_y \cdot y_t + f_t$$

$$f_x = t^2 \cdot \cos y + 2xy \cdot \ln t$$

$$x_t = 2t$$

$$f_y = -t^2 \cdot x \cdot \sin y + x^2 \cdot \ln t$$

$$y_t = \cos t$$

$$f_t = 2t \cdot x \cos y + \frac{x^2 y}{t}$$

άρα

$$\frac{df}{dt} = (t^2 \cdot \cos y + 2xy \ln t)\, 2t + (x^2 \ln t - t^2 \cdot x \cdot \sin y) \cos t + 2tx\cos y + \frac{x^2 y}{t}$$

$$\Leftrightarrow \frac{df}{dt} = 2t^3 \cos y + 4xyt \ln t + x^2 \cdot \ln t \cos t + (-t^2 x \sin y \cos t) + 2xt \cos y + \frac{x^2 y}{t}$$

$$\Leftrightarrow \frac{df}{dt} = (2t^3 + 2xt)\cos y + (4xyt + x^2 \cos t) \ln t - t^2 \cdot x \cdot \sin y \cdot \cos t + \frac{x^2 y}{t}$$

ως προς t μόνο

$$\frac{df}{dt} = (2t^3 + 2t^3)\cos(\sin t) + (4t^3 \cdot \sin t + t^4 \cdot \cos t) \ln t -$$

$$- t^4 \cdot \sin(\sin t) \cdot \cos t + t^3 \cdot \sin t.$$

14) Έστω $\dfrac{(x-y)dx+(x+y)dy}{(x^2+y^2)^u}$. Να προσδιοριστεί το u, έτσι ώστε να υπάρχει συνάρτηση, η οποία να έχει ολικό διαφορικό α' τάξης των παραπάνω παράσταση.

<u>Λύση</u>

$$\frac{(x-y)dx+(x+y)dy}{(x^2+y^2)^u} = \frac{x-y}{(x^2+y^2)^u}dx + \frac{x+y}{(x^2+y^2)^u}dy$$

Έστω $P(x,y) = \dfrac{x-y}{(x^2+y^2)^u}$ και $Q(x,y) = \dfrac{x+y}{(x^2+y^2)^u}$.

Πρέπει $P_y = Q_x$

$$P_y = \frac{-(x^2+y^2)^u-(x-y)u(x^2+y^2)^{u-1}\cdot 2y}{(x^2+y^2)^{2u}}$$

$$Q_x = \frac{(x^2+y^2)^u-(x+y)u(x^2+y^2)^{u-1}\cdot 2x}{(x^2+y^2)^{2u}}$$

$$P_y = Q_x \Leftrightarrow \frac{-(x^2+y^2)^u-(x-y)u(x^2+y^2)^{u-1}\cdot 2y}{\cancel{(x^2+y^2)^{2u}}} = \frac{(x^2+y^2)^u-(x+y)u(x^2+y^2)^{u-1}2x}{\cancel{(x^2+y^2)^{2u}}}$$

$$\Leftrightarrow -\cancel{2}(x^2+y^2)^u-(x-y)u(x^2+y^2)^{u-1}\cdot\cancel{2}y = -(x+y)u(x^2+y^2)^{u-1}\cdot\cancel{2}x$$

$$\Leftrightarrow u(x+y)\cancel{(x^2+y^2)^{u-1}}\cdot x = \cancel{(x^2+y^2)^u}+u(x-y)\cancel{(x^2+y^2)^{u-1}}\cdot y$$

$$\Leftrightarrow \frac{u(x+y)x}{x^2+y^2} = 1+\frac{u(x-y)y}{x^2+y^2}$$

$$\Leftrightarrow \frac{ux^2+uyx-x^2-y^2-uxy+uy^2}{x^2+y^2}=0$$

$$\Leftrightarrow \frac{\cancel{(x^2+y^2)}(u-1)}{\cancel{x^2+y^2}}=0 \Leftrightarrow u-1=0 \Leftrightarrow \boxed{u=1}$$

15) Να δείξετε ότι η παρακάτω συνάρτηση ικανοποιεί το θεώρημα Euler: $f(x,y) = \dfrac{x^2+y^2}{x^2-y^2}$.

Λύση

$$f(\lambda x,\lambda y) = \frac{(\lambda x)^2+(\lambda y)^2}{(\lambda x)^2-(\lambda y)^2} = \frac{\lambda^2(x^2+y^2)}{\lambda^2(x^2-y^2)} = \lambda^0 \cdot f(x,y)$$

άρα η f είναι ομογενής βαθμού $n=0$.

Θα πρέπει να δείξουμε ότι: $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 0 \cdot f = 0$.

$$\frac{\partial f}{\partial x} = \frac{2x(x^2-y^2)-(x^2+y^2)2x}{(x^2-y^2)^2} = \frac{\cancel{2x^3}-2xy^2-\cancel{2x^3}-2xy^2}{(x^2-y^2)^2} = \frac{-4xy^2}{(x^2-y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y(x^2-y^2)+(x^2+y^2)2y}{(x^2-y^2)^2} = \frac{2yx^2-\cancel{2y^3}+2yx^2+\cancel{2y^3}}{(x^2-y^2)^2} = \frac{4x^2y}{(x^2-y^2)^2}$$

άρα

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{-4x^2y^2}{(x^2-y^2)^2} + \frac{4x^2y^2}{(x^2-y^2)^2} = 0$$

δηλαδή η $f(x,y) = \dfrac{x^2+y^2}{x^2-y^2}$ ικανοποιεί το θεώρημα Euler.

—

16) Έστω $x = r\cdot\cos\theta$, $y = r\cdot\sin\theta$. Ζητείται $\dfrac{\partial r}{\partial x}$, $\dfrac{\partial \theta}{\partial x}$.

<u>Λύση</u>

$$(1)\left\{\begin{array}{l} x = g(r,\theta) \\ y = w(r,\theta) \end{array}\right\}$$

Υποθέτουμε ότι ο μετασχηματισμός αντιστρέφεται:

$r = r(x,y)$ και $\theta = \theta(x,y)$

Παραγωγίζοντας την (1) ως προς y:

$$\left\{\begin{array}{l} \dfrac{\partial x}{\partial y} = \dfrac{\partial g}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial g}{\partial \theta}\dfrac{\partial \theta}{\partial y} \\[2ex] \dfrac{\partial y}{\partial y} = \dfrac{\partial w}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial w}{\partial \theta}\dfrac{\partial \theta}{\partial y} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} g_r\cdot r_y + g_\theta\cdot\theta_y = 0 \\ w_r\cdot r_y + w_\theta\cdot\theta_y = 1 \end{array}\right\}$$

$g_r = \cos\theta$, $g_\theta = -r\cdot\sin\theta$, $w_r = \sin\theta$, $w_\theta = r\cdot\cos\theta$, <u>άρα:</u>

$$\left\{\begin{array}{l} \cos\theta\cdot r_y - r\cdot\sin\theta\cdot\theta_y = 0 \\ \sin\theta\cdot r_y + r\cdot\cos\theta\cdot\theta_y = 1 \end{array}\right\}$$

$$D = \begin{vmatrix} \cos\theta & -r\cdot\sin\theta \\ \sin\theta & r\cdot\cos\theta \end{vmatrix} = r\cdot\cos^2\theta + r\cdot\sin^2\theta = r$$

$$r_y = \begin{vmatrix} 0 & -r\cdot\sin\theta \\ 1 & r\cdot\cos\theta \end{vmatrix} \overset{:D}{=} \frac{\cancel{r}\cdot\sin\theta}{\cancel{r}} \Leftrightarrow \boxed{r_y = \sin\theta}$$

$$\theta_y = \begin{vmatrix} \cos\theta & 0 \\ \sin\theta & 1 \end{vmatrix} \overset{:D}{=} \frac{\cos\theta}{r} \Leftrightarrow \boxed{\theta_y = \frac{\cos\theta}{r}}$$

—

17) Έστω ο μετασχηματισμός $x = g(u,v) = \frac{v^2-u^2}{2}$ και $y = w(u,v) = uv$.
Ζητούνται $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$.

Λύση

Υποθέτουμε ότι $u = u(x,y)$ και $v = v(x,y)$.

Παραγωγίζοντας τις σχέσεις x, y ως προς y, έχουμε

$$\left\{\begin{aligned} \frac{\partial x}{\partial y} &= \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial y} &= \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} -u\cdot u_y + v\cdot v_y &= 0 \\ v\cdot u_y + u\cdot v_y &= 1 \end{aligned}\right\}$$

$$D = J = \begin{vmatrix} -u & v \\ v & u \end{vmatrix} = -u^2 - v^2$$

$$u_y = \frac{\begin{vmatrix} 0 & v \\ 1 & u \end{vmatrix}}{J} = \frac{v}{-u^2-v^2} \Leftrightarrow \boxed{u_y = \frac{-v}{u^2+v^2}}$$

$$v_y = \frac{\begin{vmatrix} -u & 0 \\ v & 1 \end{vmatrix}}{J} = \frac{-u}{-u^2-v^2} \Leftrightarrow \boxed{v_y = \frac{u}{u^2+v^2}}$$

---

18) Έστω $f(x,y,z) = xyz + x + y - z = 0$. Ποιες οι z_x, z_y και να εξεταστεί αν ισχύει το θεώρημα ύπαρξης στο $O(0,0,0)$.

Λύση

Για να ισχύει το θεώρημα ύπαρξης στο O, πρέπει:

I) $f(0,0,0) = 0 \Rightarrow f(0,0,0) = 0\cdot0\cdot0 + 0 + 0 - 0 = 0$ (ισχύει)

II) $f_x, f_y, f_z \rightarrow$ συνεχείς

$$\left.\begin{cases} F_y = xz+1 \\ F_z = xy-1 \end{cases}\right\}$$ συνεχείς

III) $F_z(0,0,0) \neq 0 \Rightarrow F_z(0,0,0) = 0\cdot 0 - 1 = -1 \neq 0$

άρα ισχύει στο $(0,0,0)$ το θεώρημα ύπαρξης, δηλαδή υπάρχει μοναδι[κή] διαφορίσιμη συνάρτηση $z = f(x,y)$, ώστε:

I) $z_0 = f(0,0)$

II) $F(x,y,f(x,y)) = 0$

III) $z_x = -\dfrac{F_x}{F_z} = \dfrac{-(yz+1)}{xy-1}$

$z_y = -\dfrac{F_y}{F_z} = \dfrac{-(xz+1)}{xy-1}$

19) Έστω $\left.\begin{cases} F(x,y,u,v) = u^2+v^2-x^2-y=0 \\ G(x,y,u,v) = u+v-x^2+y=0 \end{cases}\right\}$. Να εξεταστεί αν μπορούν να λυθούν ως προς u,v και να βρεθούν οι u_x, u_y, v_x, v_y. σημείο $P_0(2,1,1,2)$.

<u>Λύση</u>

...σου θεωρία)

<u>Εύρεση των u_x, u_y</u>

$$\left.\begin{matrix} dF=0 \\ dG=0 \end{matrix}\right\} \Leftrightarrow \left.\begin{cases} 2u\,du + 2v\,dv = 2x\,dx + dy \\ du+dv = 2x\,dx - dy \end{cases}\right\} \cdot(-2v)$$

$$\Rightarrow \left.\begin{cases} 2u\,du + 2v\,dv = 2x\,dx + dy \\ -2v\,du - 2v\,dv = -4vx\,dx - 2v\,dy \end{cases}\right\} (+)$$

$$(2u-2v)\,du = (2x-4vx)\,dx + (1-2v)\,dy$$

$$du = \frac{2x-4vx}{2u-2v}dx + \frac{1-2v}{2u-2v}dy$$

όμως $du = u_x dx + u_y dy$, άρα:

$$u_x = \frac{x-2vx}{u-v} \quad \text{και} \quad u_y = \frac{1-2v}{2(u-v)}$$

20) Να αναπτυχθεί η συνάρτηση $f(x,y) = x^3+y^3+xy^2$ σε δυνάμεις του $(x-1)$ και $(y-2)$.

Λύση

Αφού η f είναι μέχρι 3ου βαθμού, θα έχει μέχρι και 4η παράγωγο, δηλαδή $n=3$, γιατί μετά οι όροι μηδενίζονται.

$$f(x,y) = f(1,2) + \frac{(x-1)}{1!}f_x\Big|_{(1,2)} + \frac{(y-1)}{1!}f_y\Big|_{(1,2)} + \frac{(x-1)^2}{2!}f_{xx}\Big|_{(1,2)} + \frac{(y-2)^2}{2!}f_{yy}\Big|_{(1,2)}$$

$$\ldots + (x-1)(y-2)f_{xy}\Big|_{(1,2)} + \frac{(x-1)^3}{3!}f_{xxx}\Big|_{(1,2)} + \frac{(y-2)^3}{3!}f_{yyy}\Big|_{(1,2)} + \ldots$$

$$\ldots + \frac{3(x-1)^2(y-2)f_{xxy}\Big|_{(1,2)}}{3!} + \frac{3(x-1)(y-2)^2 f_{yyx}\Big|_{(1,2)}}{3!}$$

$$f(x,y) = x^3+y^3+xy^2 \xrightarrow{(1,2)} f(1,2) = 13$$

$$f_x = 3x^2+y^2 \xrightarrow{(1,2)} f_x\Big|_{(1,2)} = 7$$

$$f_y = 3y^2+2xy \xrightarrow{(1,2)} f_y\Big|_{(1,2)} = 16$$

$$f_{xx} = 6x \xrightarrow{(1,2)} f_{xx}\Big|_{(1,2)} = 6$$

$$f_{xxx} = 6 \xrightarrow{(1,2)} f_{xxx}\Big|_{(1,2)} = 6$$

$$f_{yy} = 6y+2x \xrightarrow{(1,2)} f_{yy}\Big|_{(1,2)} = 14$$

$$f_{yyy} = 6 \xrightarrow{(1,2)} f_{yyy}\Big|_{(1,2)} = 6$$

$$f_{xy} = 2y \xrightarrow{(1,2)} f_{xy}\Big|_{(1,2)} = 4$$

$$f_{yyx} = 2 \xrightarrow{(1,2)} f_{yyx}\Big|_{(1,2)} = 2$$

$$f_{xxy} = 0 \xrightarrow{(1,2)} f_{xxy}\Big|_{(1,2)} = 0$$

άρα

$$f(x,y) = 13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 7(y-2)^2 + 4(x-1)(y-2) + \ldots$$
$$\ldots + (x-1)^3 + (y-2)^3 + (x-1)(y-2)^2$$

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21) Να αναπτυχθεί σε σειρά McLaurin η $f(x,y) = e^x \cdot \sin y$ για $n=3$.

Λύση

$f(x,y) = e^x \cdot \sin y \quad , \quad (x-0), (y-0), n=3.$

$$f(x,y) = f(0,0) + x \cdot f_x|_{(0,0)} + y\, f_y|_{(0,0)} + \frac{x^2}{2!} f_{xx}|_{(0,0)} + \frac{y^2}{2!} f_{yy}|_{(0,0)} + \ldots$$
$$\ldots + xy \cdot f_{xy}|_{(0,0)} + \frac{x^3}{3!} f_{xxx}|_{(0,0)} + \frac{y^3}{3!} f_{yyy}|_{(0,0)} + \ldots$$
$$\ldots + \frac{3x^2y\, f_{xxy}|_{(0,0)}}{3!} + \frac{3xy^2 f_{yyx}|_{(0,0)}}{3!}$$

$f(x,y) = e^x \cdot \sin y \xrightarrow{(0,0)} f(0,0) = e^0 \cdot \sin 0 = 0$

$f_x = e^x \cdot \sin y \xrightarrow{(0,0)} f_x|_{(0,0)} = 0$

$f_{xx} = f_{xxx} = e^x \cdot \sin y \xrightarrow{(0,0)} f_{xx}|_{(0,0)} = f_{xxx}|_{(0,0)} = 0$

$f_y = e^x \cdot \cos y \xrightarrow{(0,0)} f_y|_{(0,0)} = 1$

$f_{yy} = -e^x \cdot \sin y \xrightarrow{(0,0)} f_{yy}|_{(0,0)} = 0$

$f_{yyy} = -e^x \cdot \cos y \xrightarrow{(0,0)} f_{yyy}|_{(0,0)} = -1$

$f_{xy} = e^x \cdot \cos y \xrightarrow{(0,0)} f_{xy}|_{(0,0)} = 1$

$f_{xxy} = e^x \cdot \cos y \xrightarrow{(0,0)} f_{xxy}|_{(0,0)} = 1$

$f_{yyx} = -e^x \cdot \sin y \xrightarrow{(0,0)} f_{yyx}|_{(0,0)} = 0$

άρα: $f(x,y) = y + xy - \frac{y^3}{6} + \frac{x^2y}{2}$

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22) Να υπολογιστούν οι ακραίες τιμές της συνάρτησης $y=y(x)$ που ορίζεται από την εξίσωση $f(x,y)=x^2+y^2-4x+y-2=0$. (10)

Λύση

Η $f(x,y)$ είναι πεπλεγμένη. Θέλω ακρότατα της $y(x)$.

$$\left\{\begin{array}{l} f(x,y)=0 \\ f_x=0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x^2+y^2-4x+y-2=0 \\ 2x-4=0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} 4+y^2-8+y-2=0 \\ x=2 \end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{l} y^2+y-6=0 \\ x=2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} y=2 \\ x=2 \end{array}\right\} \text{ ή } \left\{\begin{array}{l} y=-3 \\ x=2 \end{array}\right\}$$

$$f_y=2y+1 \Rightarrow f_y\big|_{(2,2)}=5\neq 0$$

$$f_y\big|_{(2,-3)}=-5\neq 0$$

$$f_{xx}=2 \Rightarrow f_{xx}\big|_{(2,2)}=f_{xx}\big|_{(2,-3)}=2$$

$$f_{yy}=2 \Rightarrow f_{yy}\big|_{(2,2)}=f_{yy}\big|_{(2,-3)}=2$$

$$f_{xy}=f_{yx}=0 \Rightarrow f_{xy}\big|_{(2,2)}=f_{xy}\big|_{(2,-3)}=f_{yx}\big|_{(2,2)}=f_{xy}\big|_{(2,-3)}=0$$

$$H=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}=4$$ και για τα δύο σημεία

Σημείο P_{0_1}

$f_y\cdot f_{xx}=5\cdot 2=10>0$, $H=4>0$

άρα το P_{0_1} είναι τοπικό μέγιστο, δηλαδή $f(2)=f_{max}=2$.

Σημείο P_{0_2}

$f_y\cdot f_{xx}=-5\cdot 2=-10<0$, $H=4>0$

άρα το P_{0_2} είναι τοπικό ελάχιστο, δηλαδή $f(2)=f_{min}=-3$.

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